MATH 2020 Tutorial 4

① $\int_1^e \int_1^{e^2} \int_1^{e^3} \frac{1}{xyz}\, dx\, dy\, dz$

$= \int_1^e \int_1^{e^2} \frac{1}{yz} \ln x \Big|_1^{e^3} dy\, dz$

$= \int_1^e \int_1^{e^2} \frac{3}{yz}\, dy\, dz$

$= \int_1^e \frac{3}{z} \ln y \Big|_1^{e^2} dz = \int_1^e \frac{3}{z} \cdot 2\, dz = 6 \ln z \Big|_1^e = 6.$ #

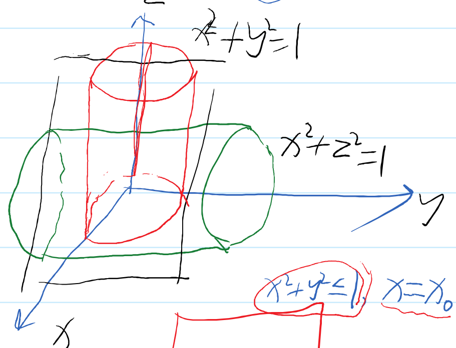
② The region common to the interiors of the cylinders $x^2+y^2=1$, $x^2+z^2=1$.

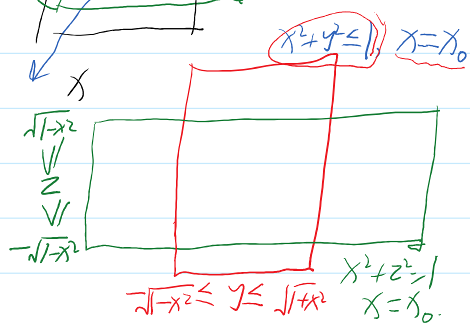
Ans: $\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\, dy\, dx$

$= \int_{-1}^{1} 2\sqrt{1-x^2} \cdot 2\sqrt{1-x^2}\, dx$

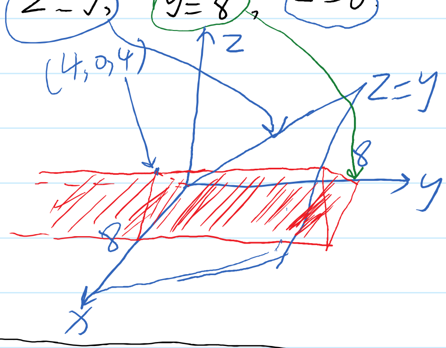
$= \int_{-1}^{1} 4 - 4x^2\, dx$

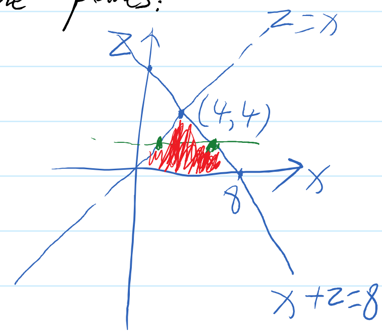
$= 4x - \frac{4}{3}x^3 \Big|_{-1}^{1} = 2\left[4 - \frac{4}{3}\right] = \frac{16}{3}.$



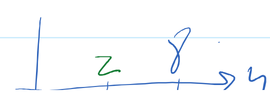


③ Find the volume of the region bounded by the planes: $z=x$, $x+z=8$, $z=y$, $y=8$, $z=0$

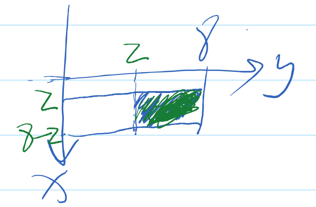




$\int_0^4 \int^{8-z} \int^{8} \ldots dy\ \ldots dz$



$$\int_0^4 \left[\int_2^{8-z} \int_2^{8} dy\, dx\right] dz.$$



$$= \int_0^4 (8-z-z)(8-z)\,dz = \int_0^4 64 - 24z + 2z^2\,dz$$

$$= 64z - 12z^2 + \frac{2}{3}z^3 \Big|_0^4 = 256 - 192 + \frac{128}{3} = \frac{320}{3}$$

④ Find the average value of $F(x,y,z) = x^2 + 9$ over the cube in the first octant bounded by the coordinate planes and the planes $x=2$, $y=2$, $z=2$.

Ans: $$\bar{F} = \frac{1}{2^3} \int_0^2 \int_0^2 \int_0^2 (x^2+9)\,dz\,dy\,dx$$

$$= \frac{1}{8} \cdot \int_0^2 2\times 2\times (x^2+9)\,dx$$

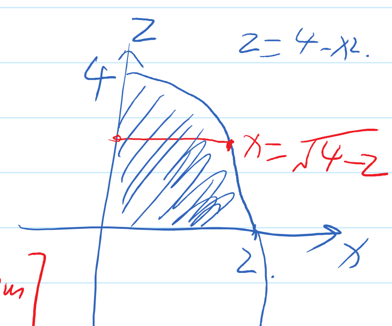
$$= \frac{1}{2} \int_0^2 x^2+9\,dx = \frac{1}{2}\left[\frac{1}{3}x^3 + 9x\right]_0^2$$

$$= \frac{1}{2}\left(\frac{8}{3} + 18\right) = \frac{1}{2} \cdot \frac{62}{3} = \frac{31}{3}.$$ #.

⑤ $$\int_0^2 \int_0^{4-x^2} \left[\int_0^x \frac{\sin 2z}{4-z}\,dy\right] dz\,dx$$

$$= \int_0^2 \int_0^{4-x^2} \frac{x \cdot \sin 2z}{4-z}\,dz\,dx.$$

$$= \int_0^4 \left[\int_0^{\sqrt{4-z}} \frac{x \sin 2\cdot z}{4-z}\,dx\right] dz$$ [Fubini's Thm]



$$= \int_0^4 \frac{\sin 2z}{4-z} \cdot \frac{1}{2}x^2 \Big|_0^{\sqrt{4-z}} dz = \int_0^4 \frac{\sin 2z}{4-z} \cdot \frac{1}{2}(4-z)\,dz$$

$$= \int_0^4 \frac{\sin 2z}{2}\,dz = -\frac{\cos 2z}{4}\Big|_0^4 = -\frac{\cos 8 - 1}{4}$$

⑥ Solve for a.

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15}$$

$$\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dz\,dy\,dx = \frac{4}{15}$$

$$= \int_0^1 \int_0^{4-a-x^2} (4-x^2-y-a)\,dy\,dx$$

$$= \int_0^1 (4-a-x^2)y - \frac{1}{2}y^2 \Big|_0^{4-a-x^2} dx$$

$$= \int_0^1 (4-a-x^2)^2 - \frac{1}{2}(4-a-x^2)^2 dx = \int_0^1 \frac{1}{2}((4-a)-x^2)^2 dx$$

$$= \int_0^1 \frac{1}{2}(4-a)^2 - 2(4-a)x^2 + x^4\,dx$$

$$= \frac{1}{2}\left[(4-a)^2 x - (4-a)\cdot\frac{2}{3}x^3 + \frac{1}{5}x^5\right]\Big|_0^1$$

$$= \frac{1}{2}\left[(4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5}\right] = \frac{4}{15}$$

$$(4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5} = \frac{8}{15}$$

$$15(4-a)^2 - 10(4-a) + 3 = 8$$

$$15(4-a)^2 - 10(4-a) - 5 = 0$$

$$3(4-a)^2 - 2(4-a) - 1 = 0$$

$$[3(4-a)+1][(4-a)-1] = 0$$

$$4-a = -\frac{1}{3} \quad \text{or} \quad 4-a = 1$$

$$a = \boxed{\frac{13}{3}} \quad \text{or} \quad a = \boxed{3}$$

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$$= \int_1^e \int_1^{e^2} \frac{3}{yz}\,dy\,dz = \int_1^e \frac{3}{z} \ln y \Big|_1^{e^2} dz$$

$$= \int_1^e \frac{3}{z}\cdot 2\,dz = 6\cdot \ln z \Big|_1^e = 6.$$

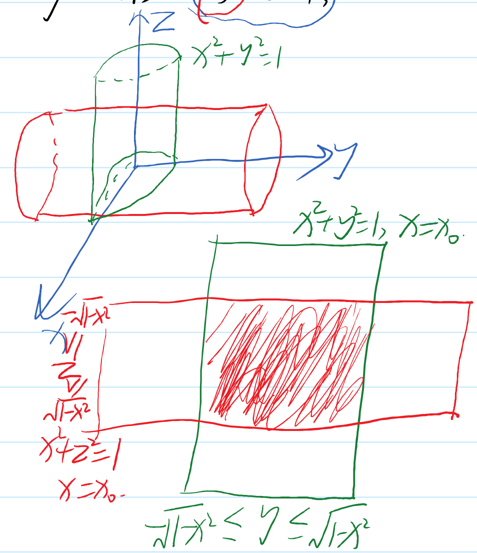
② The region common to the interiors of the cylinders $x^2+y^2=1$, $x^2+z^2=1$.

Ans: The volumn is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\,dy\,dx$$

$$=\int_{-1}^{1} 2\sqrt{1-x^2}\cdot 2\sqrt{1-x^2}\,dx$$

$$=\int_{-1}^{1} 4-4x^2\,dx = 4x-\frac{4}{3}x^3\Big|_{-1}^{1}=\frac{16}{3}$$



③ Find the volumn of the region bounded by the planes $z=x$, $x+z=8$, $z=y$, $y=8$, $z=0$

no y no y no y

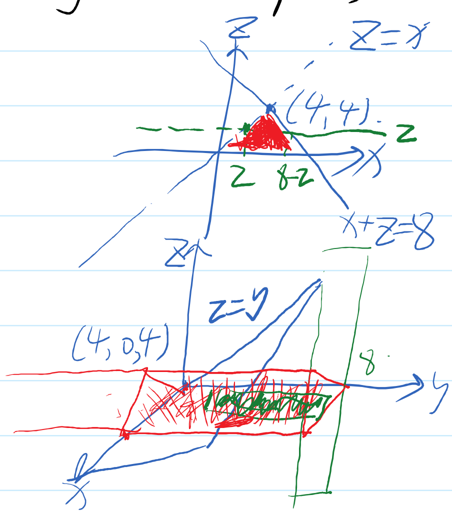
Ans: $$\int_0^4\int_z^{8-z}\int_z^{8} dy\,dx\,dz$$

$$=\int_0^4 (8-z-z)(8-z)\,dz$$

$$=\int_0^4 (8-2z)(8-z)\,dz$$

$$=\int_0^4 64-24z+2z^2\,dz$$

$$=64z-12z^2+\frac{2}{3}z^3\Big|_0^4=\frac{320}{3}\ \#.$$



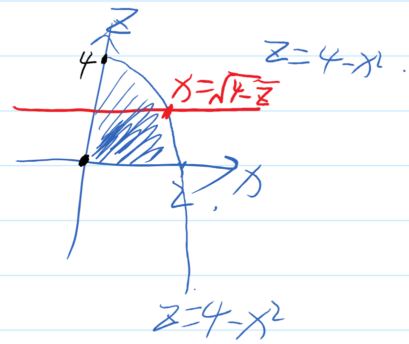
④ Find the average value of $F(x,y,z)=x^2+9$ over the cube in the first octant bounded by the coordinate planes and the planes $x=2$, $y=2$, $z=2$.

Ans: $$\bar{F}_D=\frac{\int_D F}{|D|}=\frac{1}{2\times2\times2}\int_0^2\int_0^2\int_0^2 x^2+9\,dz\,dy\,dx.$$

$$=\frac{1}{8}\int_0^2 (x^2+9)\cdot 2\times 2\,dx$$

$$= \frac{1}{2}\int_0^2 x^2+9\,dx = \frac{1}{2}\left(\frac{1}{3}x^3+9x\right)\Big|_0^2 = \frac{31}{3} \quad \#.$$

(5) $$\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx$$



$$=\int_0^2\int_0^{4-x^2}(x-0)\cdot\frac{\sin 2z}{4-z}\,dz\,dx$$

$$=\int_0^2\int_0^{4-x^2}\frac{x\cdot\sin 2z}{4-z}\,dz\,dx$$

$$=\int_0^4\int_0^{\sqrt{4-z}}\frac{x\sin 2z}{4-z}\,dx\,dz \quad \text{[Fubini's Thm]}$$

$$=\int_0^4\frac{\sin 2z}{4-z}\cdot\frac{1}{2}x^2\Big|_0^{\sqrt{4-z}}\,dz$$

$$=\int_0^4\frac{\sin 2z}{4-z}\cdot\frac{1}{2}\cdot(4-z)\,dz$$

$$=\int_0^4\frac{\sin 2z}{2}\,dz = -\frac{\cos 2z}{4}\Big|_0^4 = -\frac{\cos 8-1}{4} \quad \#$$

(6) Solve for a.

$$\int_0^1\int_0^{4-a-x^2}\int_a^{4-x^2-y}dz\,dy\,dx = \frac{4}{15}.$$

$$=\int_0^1\int_0^{4-a-x^2}(4-x^2-y-a)\,dy\,dx$$

$$=\int_0^1 (4-x^2-a)y-\frac{1}{2}y^2\Big|_0^{4-a-x^2}dx.$$

$$=\int_0^1 (4-a-x^2)^2-\frac{1}{2}(4-a-x^2)^2\,dx$$

$$=\int_0^1\frac{1}{2}((4-a)-x^2)^2\,dx$$

$$=\frac{1}{2}\int_0^1 (4-a)^2-2(4-a)x^2+x^4\,dx$$

$$=\frac{1}{2}\left[(4-a)^2x-\frac{2}{3}(4-a)x^3+\frac{1}{5}x^5\right]\Big|_0^1$$

$$=\frac{1}{2}\left[(4-a)^2-\frac{2}{3}(4-a)+\frac{1}{5}\right]=\frac{4}{15}.$$

$$= \frac{1}{2}\left[(4-a)^2 - \frac{2}{3}(4-a) + \frac{1}{5}\right] = \frac{4}{15}.$$

$$15(4-a)^2 - 10(4-a) + 3 = 8$$

$$15(4-a)^2 - 10(4-a) - 5 = 0$$

$$3(4-a)^2 - 2(4-a) - 1 = 0 \qquad 4-a=t$$

$$[3(4-a)+1][(4-a)-1] = 0$$

$$4-a = -\frac{1}{3} \text{ or } 4-a=1$$

$$a = \frac{13}{3} \text{ or } a=3.$$

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